Nonlinear Resonance in Regular, Random and Small-World Networks

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Abstract

We report our investigation on the features of nonlinear resonance in regular, random and small-world networks of Duffing oscillator with bidirectional coupling. In these networks, resonance occurs when the frequency of the external periodic force or the coupling strength δ is varied. In a regular network for fixed values of the control parameters the average amplitude varies and reaches a saturation with increase in the number of connectivity. In random and small-world networks the average response amplitude either increases or decreases with the parameter p(the measure of the randomness of the connectivity) depending upon the coupling strength. The average response amplitude can be controlled by varying the control parameters p and δ for fixed values of the amplitude and frequency of the external periodic signal. Also, we present the variation of distribution of response amplitudes of units of networks near and far before and far after resonance.

Keywords: Nonlinear resonance, Duffing oscillator, Networks, Bidirectional coupling, Hysteresis.

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I INTRODUCTION

Nonlinear systems are capable of displaying variety of regular and irregular dynamics. Resonance is one of the fundamental phenomena exhibited by linear and nonlinear systems. In a dynamical system driven by an external periodic force, when the frequency of the force is varied, in a typical case, the amplitude of oscillation increases and reaches a significantly large value at a frequency and then decreases. Realization of maximum amplitude is called resonance [1–3]. The effect of resonance is to produce large amplitude oscillation. Resonance in a nonlinear system is different from a linear system. Particularly, hysteresis and jump in resonance curve can be realized in nonlinear systems. Resonance occurs in many branches of Physics, Engineering and biology. It can be utilized to produce vibrations with require frequency and to choose a specific frequency from multi-frequency oscillation. Some of the notable applications of resonance include radio tuning, quartz time-piece, voltage and current application, resonant electronic circuits, resonant vibrations in musical instruments and shattering gall stones in patients using ultrasound.

Resonance can be realized by an additive and multiplicative periodic force [1–3], an external noise [4], a chaotic signal [5] and a high-frequency periodic force [6, 7]. In the present report we restrict our investigation resonance on resonance induced by additive external periodic force. In recent years a great deal of research activity has been devoted for the study of dynamics exhibited by networks. Arrays of coupled systems forming networks are used to model biological oscillators [8, 9], excitable media [9], neural networks [10, 11] and genetic networks [12–14]. Networks of flux gate magnetometers are useful to enhance the utility and sensitivity of nonlinear sensors such as magnetometers and ferroelectric detectors for electric fields [15, 16]. Because the networks carry information from one part of it from another and become a source of amplification of information. It is important to study the role of different kinds of couplings and connectivity structure in networks. There are different kinds of networks [9]. In general networks are classified into directed networks, unidirectional networks, random networks and small-world networks. Study of features of various nonlinear phenomena in different types of networks is of great significance. Very recently, the features of nonlinear resonance and stochastic resonance in unidirectionally coupled oscillators with first system alone driven by an external force are reported [17–19].

In this paper we investigate the features of nonlinear resonance in regular, random and

small-world networks. That is, we consider a network system of N units with (i) regular connectivity, (ii) random connectivity and (iii) both regular and random connectivities (small-world networks). The units in each network is chosen as the ubiquitous Duffing oscillator. The coupling between two units is bidirectional. Resonance occurs in all the three networks. We denote Q_i as the response amplitude of i^{th} unit of a network. $Q_i^{'s}$ are distributed over an interval. Therefore, we compute average response amplitude $< Q >$. In the regular network, the effect of the coupling constant, the frequency (ω) of the external periodic force and the number of coupling on average response amplitude $< Q >$ are analyzed. In the random and small-world networks in addition to the parameters (ω) and (δ) there is another control parameter p. This parameter describes the degree of randomness in the connectivity. We explore the dependence on p and (δ). In these two networks resonance occurs when the coupling strength (δ) is varied while $< Q >$ varies monotonically with p. the width of probability distribution of Q_i is wide near resonance and it decreases as the control parameter is varied on either side of resonance.

II DESCRIPTION OF THE NETWORK MODELS AND SETUP

A network of systems essentially consists of number of dynamical systems connected in pairs. The connectivity is termed as **edges**. The individual systems are called **nodes** or **units** [9]. In a regular network of n units each unit is connected to m nearest neighbours. m is usually much smaller than n. Figure (1(a)) depicts a network of 12 units with each unit connected to adjacent two nearest neighbours ($m = 2$). That is i^{th} unit is connected to $i - 2, i - 1, i$ and $(i + 1)^{th}$ units. We choose the coupling as bidirectional. This means when the unit i is connected with $i + 1$ then information flows from the i^{th} unit to $(i+1)$ and also from $(i + 1)$ to i^{th} unit. In n units network with m connectivity if $i - j < 0$, $j = 1, 2, ...m$ then $i - j = n + i - j$. If $i + j > n$ then $i + j = i + j$ (mod n).

In a regular network when the connectivity of a fraction of number of units is disabled and then connected to randomly chosen other units the resultant network is the Strogatz-Walts small-world network. That is, we require each edge randomly with probability p. Suppose there are say 12 units and $p = 0.25$. For an i^{th} unit generate a uniform random number between 0 to 1. If number becomes less than 0.25 leave the connection between i and $i + 1^{th}$ modes as such. Otherwise generate another integer number 1 to N from a uniform

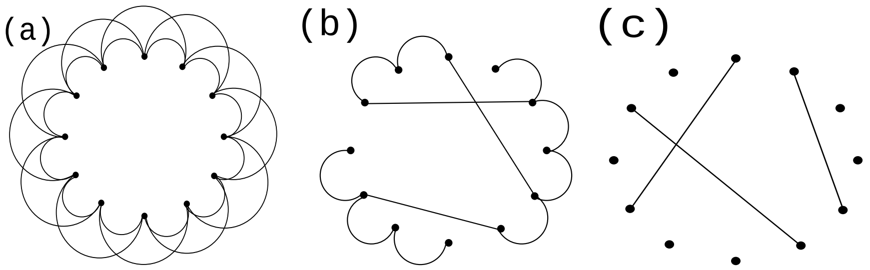

FIG. 1: (a) A regular network with nearest and next nearest neighbouring connectivity. (b) A Walts-Strogatz small-world network with nearest neighbouring regular connection and with rewriting probability p=0.25. (c) A random network with connectivity probability p=0.25.

distribution. If this number is equal to i generate another number and so on until the generated number becomes different from i. Let the generated number is j. Then connect i^{th} unit to the j^{th} unit. Repeat the above process for all units for m nearest neighbouring connectivity one end up with a small-world network. Figure (1(b)) displays a small world network with $n = 12, p = 0.25$ and $m = 1$. Note that $p = 0$ corresponds to a regular network. When $p = 1$ then all units are connected to randomly chosen units. In the third kind of network, known as random network, initially all the units are unconnected and connectivity is introduced for the units at random with probability p. An example of a random network with $n = 12$, $p = 0.25$ is shown in Fig.(1(c)). In all the above three types of networks the coupling (connectivity) between the units is bidirectional.

Consider each unit in the three types of network described above is the prototype Duffing oscillator. Then the equations of motion of the above three types of network of n Duffing oscillators is

$$\ddot{x} + d\,\dot{x} + \omega_0^2 x_i + \beta x_i^3 = f\ \cos\ (\omega t + \phi_i)\ + \frac{1}{M(i)} \sum_{j=1}^{n} \delta_{ij}\ (x_i - x_j) \tag{1}$$

where $i = 1, 2, ..., n$, ϕ_i is a random phase factor and is a set of uniformly distributed random numbers in the interval $[0, 2\pi]$ and $M(i)$ is the number of distinct units connected with the i^{th} unit.

III NUMERICAL RESULTS

The system of equations of motion (1) are numerically integrated with fourth order Runge-Kutta method with step size $(2\pi/\omega)/1000$. After leaving sufficient transient the following quantities are computed.

$$Q_{\mathrm{i,S}} = \frac{1}{NT}\int_0^{NT} x_i(t)\sin(\omega t)\,\mathrm{d}t\ , \tag{2a}$$

$$Q_{\mathrm{i,C}} = \frac{1}{NT}\int_0^{NT} x_i(t)\cos(\omega t)\,\mathrm{d}t\ , \tag{2b}$$

where $T = 2\pi/\omega$ and $N = 1000$. Next the response amplitude Q_i of i^{th} oscillator (unit) is calculated through

$$Q_i = \frac{\sqrt{Q^2_{\mathrm{i,S}} + Q^2_{\mathrm{i,C}}}}{f}\ . \tag{3}$$

Then the average value of Q_i is computed as

$$< Q >= \frac{1}{n}\sum_{i=1}^{n} Q_i. \tag{4}$$

A Regular Network

First, we consider the regular network with $m = 1$. For our numerical study we fix $d = 0.1, \omega_0^2 = 1, \beta = 2$ and $f = 0.2$. Figure (2(a)) shows the resonance curves for three fixed values of δ. For $\delta = 0$, $< Q >$ increases with increase in ω, reaches a maximum value and then suddenly jumps to a lower value. Resonance occurs at $\omega = 1.97$ at which $< Q >_{max} = 6.83$. For $\delta = -0.5$ and 0.5 resonance takes place, however, suddenly jump is suppressed. The value of ω at which resonance occurs and the corresponding value of $< Q >$ are sensitive to the value of δ. When $\delta = -0.5$ and 0.5 resonance is observed at $\omega = 1.29$ and 1.42 respectively with $< Q > = 3.81$ and 3.42 respectively. Figure (2(b)) illustrates the variation $< Q >$ with δ for several fixed values of ω. For some values of ω only one resonance occurs and for certain range of values of ω we can notice two resonances. For example with $\omega = 2.5$ there are two resonances one at $\delta = -1.65$ with $< Q >= 1.92$ while another at $\delta = 3.55$ with $< Q >= 1.52$. When the value of $|\delta|$ is greater than a certain value the motion is unbounded. In Fig.(2(c)) the range of values of $< Q >$ in the parameter space is depicted. Though the network is a regular network the quantities Q_i's are not the same. $Q_i's$ are

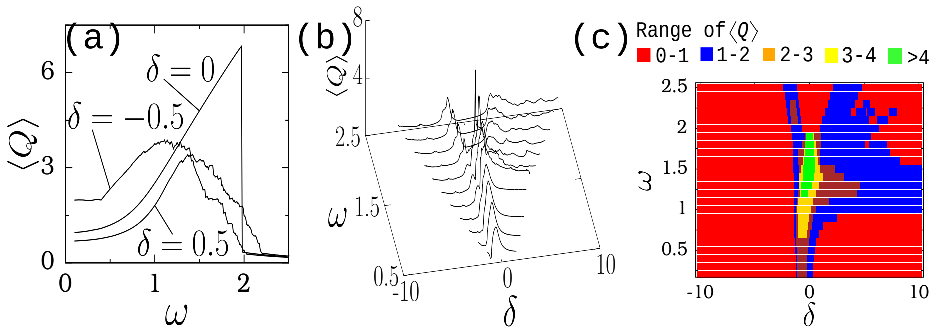

FIG. 2: The system is a regular network with $m = 1$ and $n = 1000$. (a) $< Q >$ values of ω for a few fixed values of δ (b) $< Q >$ as a function of ω and δ. (c) Plot of range of $< Q >$ in the (ω, δ) parameters plane.

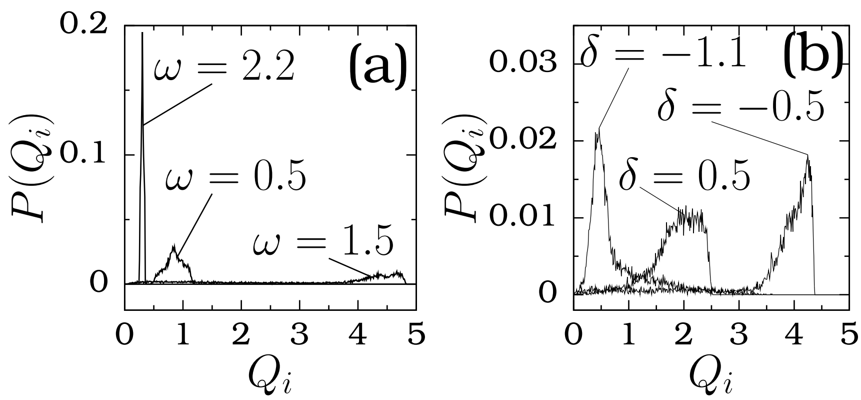

FIG. 3: P(Q) as a function of Q in the case of regular network for (a) three values of ω with $\delta = 0.5$ and (b) three values of δ with $\omega = 2.2$. Here $m = 1$.

distributed over a range. In order to know about the distribution of Q_i and its dependence on the parameters we choose $n = 10^4$ and compute $P(Q_i)$ probability distribution of Q_i. In Fig.(3(a)) $P(Q_i)$ versus Q_i is shown for three values of ω where $\delta = 0.5$. As mentioned above for this value of δ resonance occurs at $\omega = 1.42$. For $\omega = 0.5$ and 2.2 (far from resonance) is relatively narrower than that of $\omega = 1.5$ (near resonance). Similar trend is found when δ is the control parameter. In Fig.(3(b)) we can clearly see the difference between p of $\delta = 0.5$ (for which $< Q > = 3.79$) and p of $\delta = -1.1$ (with$< Q > = 0.61$) and (with $< Q > = 1.8$). Figure (4) presents the effects of the number of nearest neighbour connections m on $< Q >$ for several values of δ with $\omega = 0.5$. Depending upon the value of δ the quantity $< Q >$

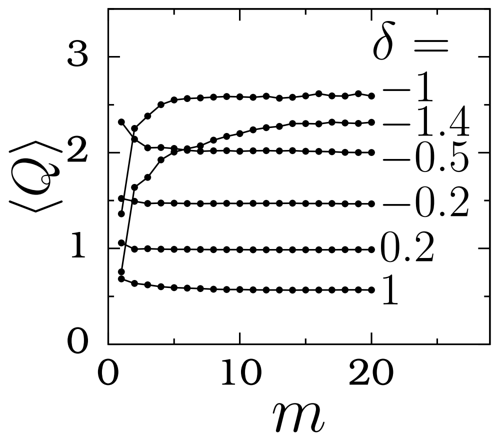

FIG. 4: Dependence of $< Q >$ with m in the regular network for $\omega = 0.5$ and for certain chosen values of δ.

either decreases or increases with m and approaches a saturation. For each value of δ and ω there is a minimum value of m to realize maximum or minimum value of $< Q >$. Increasing the value of m beyond a certain critical value cannot give a considerable change in $< Q >$.

B Small-World Network

In a small world network there are regular connections as well as random connections. Here again all the units are driven by the same external periodic force. We performed analysis with $n = 10^3$ and $m = 1$. We choose $d = 0.5, f = 0.2, \omega_0^2 = 1$ and $\beta = 0.1$. Resonance is realized when the control parameter ω and δ are varied for each fixed value of $p[0,1]$. In Fig.(5(a)) we plot $< Q >$ as a function of the coupling strength δ for four fixed values of p and for $\omega = 0.5$. For all the chosen values of p when δ is decreased from the value 0 the quantity $< Q >$ increases for a while, reaches a maximum value and then decreases. Figure (5(b)) shows the variation of $< Q >$ in the (δ, p) parameter space.

Next we explore the influence the parameter p. $p = 0$ corresponds to pure regular network while $p = 1$ for purely random network. We computed $< Q >$ for a set of values of p in the interval $[0, 1]$ for several fixed values of δ and ω. Figures (6(a)) and (6(b)) display the variation of $< Q >$ for $\omega = 0.3$ and for $\delta = -1, -0.7, -0.6, -0.57, -0.5, -0.4, -0.3$ and 0.1. In Fig.(6(a)) for $\delta = -1, -0.7, -0.6$ and -0.57 the average response amplitude on the average increases with increase in p. In Fig.(6(b)) for $\delta = -0.5$ and -0.4 on the average

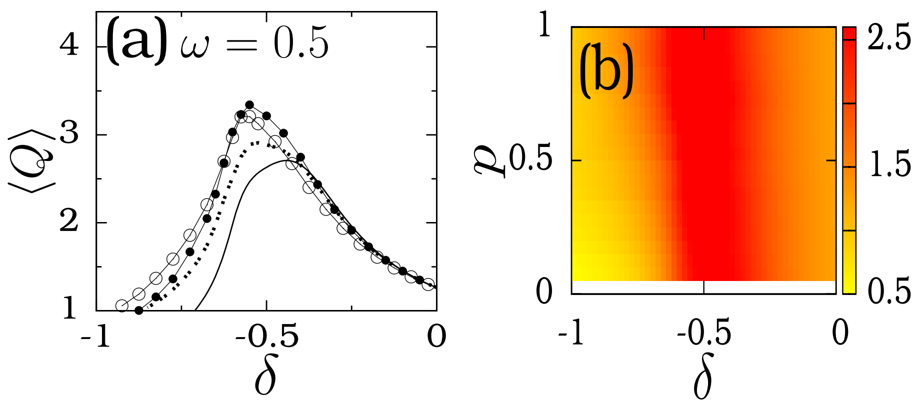

FIG. 5: $< Q >$ versus for the small-world network system for four fixed values of p with $\omega = 0.5$ the values of p for the continuous curve, dashed curve, line connected painted circles and line connected open circles are $0, 0.4, 0.6$ and 1 respectively. (b) Dependence of $< Q >$ on δ and p for $\omega = 0.5$.

$< Q >$ decreases with increase in p. For $\delta = -0.3$ and 0.1 the variation in $< Q >$ with p is almost negligible. For other values of ω also $< Q >$ increases (decreases or almost constant) with p for certain range of values of δ. The effect of p is not the same for each fixed value of δ. Figure (7) presents $P(Q_i)$ as a function of Q_i for $\delta = -1, -0.7$ and -0.5 with $\omega = 0.5$ and $p = 0.2$. For $\delta = -1, -0.7$ and -0.5 the values of $< Q >$ are $0.79, 1.33$ and 2.69 respectively. $P(Q_i)$ is very narrow for $\delta = 1$ for which is $< Q >$ relatively small. In Fig.(7) we note that $P(Q_i)$ for $\delta = -0.5$ (for which $< Q >$ is relatively higher than that of $\delta = -1$ and -0.7) is relatively wider than the other two values of considered in Fig.(7).

C Random Network

In the random network also the degree of randomness is described by the parameter p. Typical resonance occurs when ω is varied for various fixed values of δ and p. Here our focus is on the effect of p and δ on the average response amplitude $< Q >$. In Fig.(8(a)) we plotted the numerically computed $< Q >$ for a range of values of δ for several fixed values of p. Here $d = 0.1, f = 0.2, \omega_0^2 = 1, \beta = 2$ and $\omega = 0.5$. For each fixed value of p as δ increases from, for example, say, -1 then $< Q >$ increases with δ, reaches a maximum value at a value of δ and then decreases. That is, by varying the strength of the coupling of the units resonance can be realized. Further in Fig.(8) we clearly notice that for each fixed value

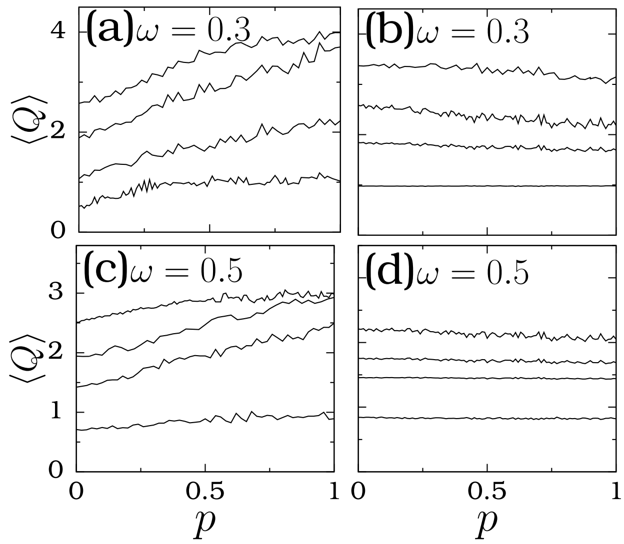

FIG. 6: Dependence of $< Q >$ on the control parameter for certain fixed values of δ for two values of ω. (a) The values of δ for the curves bottom to top are $-1, -0.7, -0.6, -0.57$. (b) the values of δ for the curves top to bottom are $-0.5, -0.4, -0.3, 0.1$. (c) The values of δ for the curves bottom to top are $-1, -0.65, -0.6, -0.5$. (d) The values of δ for the curves top to bottom are $-0.3, -0.2, -0.1, 0.5$.

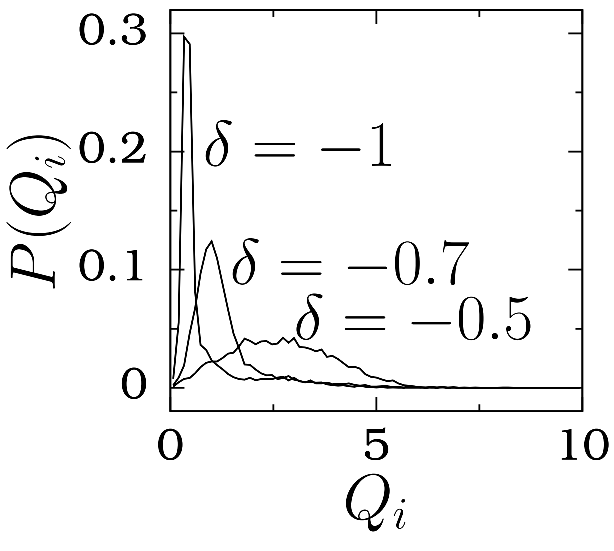

FIG. 7: $P(Q)$ versus Q for the small-world network system for $p = 0.2$, $\omega = 0.5$ and for $\delta = -1, -0.7$ and -0.5.

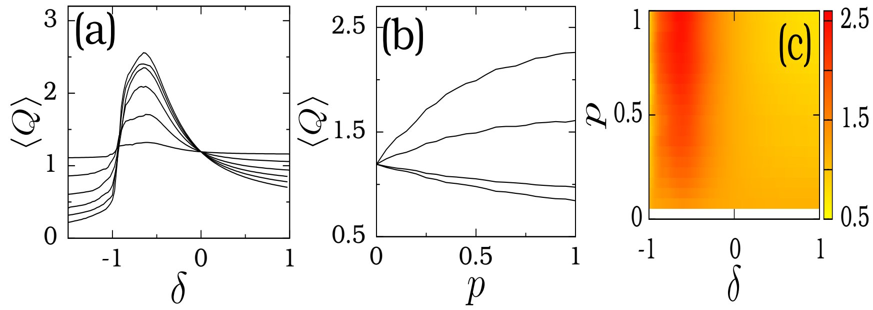

FIG. 8: (a) Plot of $< Q >$ versus δ for the random network system for several fixed values of p for ω. The values of p for the curves bottom to top are $0.05, 0.2, 0.4, 0.6, 0.8$ and 1 respectively. (b) $< Q >$ as a function of p for four fixed values of δ with $\omega = 0.5$. The values of δ for the curves top to bottom are -0.5, -0.25, 0.25 and 0.5 respectively. (c) Dependence of $< Q >$ on δ and p for $\omega = 0.5$.

of $\delta < 0$, $< Q >$ increases with increase in p. For $\delta > 0$, $< Q >$ decreases with increase in p. For $\delta > 0$, $< Q >$ decreases with increase in δ. Figure (8(b)) shows the variation of $< Q >$ with the parameter p for $\delta = -0.5, -0.25, 0.25$ and 0.5. For $\delta = -0.5$ and -0.25, $< Q >$ increases with p monotonically. There is no resonance when p is varied. $< Q >$ decreases with p for $\delta = 0.25$ and 0.5. For $\delta > 0$ the increase in the number of connectivity increases $< Q >$ while for $\delta < 0$ the increase in the number of connectivity decreases the value of $< Q >$. Figure (8(c)) depicts the variation of $< Q >$ in (δ, p) parameter space. In the random network also $Q_i's$ take a range of values. Figure (9) shows the change in $P(Q_i)$ with p for $\delta = -0.6$. As seen in Fig.(8(b)) $< Q >$ increases with p. In Fig.(9) for $P(Q_i)$, $p = 0.2$ has a sharp peak at a value of Q_i. The height of the peak decreases with increase in p. For $p = 0.2, 0.5$ and 1 the values of $< Q >$ are $1.59, 2.08$ and 2.51 respectively. That is the amplitude of the dominant peak decreases with increase in the value of $< Q >$. Fluctuation of Q_i is relatively larger for large values of $< Q >$.

IV CONCLUSION

In the present paper we studied the effect of connectivity topology of three networks on the nonlinear resonance. Each unit in the network is the Duffing oscillator driven by a periodic force. In all the networks resonance is realized when either the parameter ω

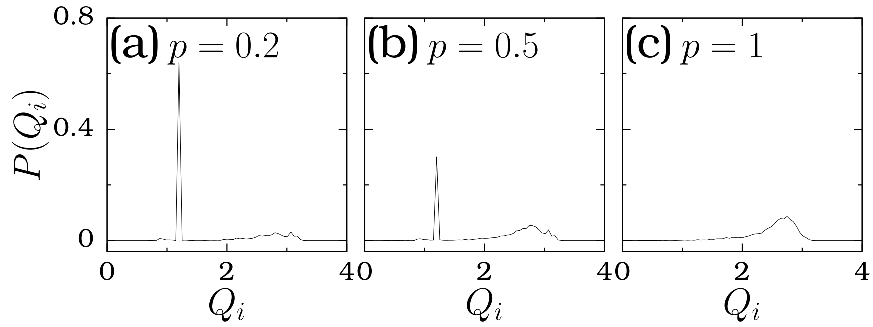

FIG. 9: Numerically computed $P(Q)$ for random network for three values of p with $\delta = -0.6$ and $\omega = 0.5$.

or the strength δ of the coupling is varied. Resonance is not realized when the degree of randomness p is varied in the small-world and random networks. However $< Q >$ is found to increase or decrease with the parameter p depending upon the values of the other parameters. The probability distribution of $< Q >$ has shown the same behavior in the three networks. In addition to nonlinear resonance other types of resonances have been found to occur in nonlinear systems. Examples of other resonances include stochastic, vibrational, coherence and ghost resonances. Investigation of features of these resonances with specific emphasis on a critical comparison of influence of connectivity structure on $< Q >$ provide better understanding of the role of connectivity on the resonance dynamics.

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